Characteristic Modes for Aperture Problems

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Abstract—A theory of characteristic modes is developed for problems consisting of two regions coupled by an aperture. The modes are derived from a weighted eigenvalue equation whose eigenfunctions define a set of real expansion functions for the equivalent magnetic current over the aperture region and whose eigenvalues are the modal aperture admittances. A modal solution is obtained for an aperture of arbitrary size and shape coupling two regions of arbitrary size and shape. The theory provides a rigorous basis for the augmentation of the Bethe-hole theory by radiation conductance terms, for its extension to apertures of larger electrical size, and for its extension to apertures in nonplanar conducting surfaces.

I. Introduction

THE THEORY OF characteristic modes has proved very useful in the analysis of electromagnetic scattering problems [1]–[4]. These modes are basically solutions to a weighted eigenvalue equation involving the impedance operator Z, which relates the surface current on a conductor to the tangential component of the incident electric field on the conductor. The modal currents are real (or equiphasal), orthogonal over the conducting surface with weight operator $\operatorname{Re} Z$, and the modal radiation fields are Hermitian orthogonal over the radiation sphere. When used in a modal solution, they give a radiation field which converges in a least-squares sense on the radiation sphere.

A similar theory of characteristic modes can be developed for the equivalent magnetic current in an aperture problem. As shown in [5], the problem of coupling from one region to another region through an aperture can be formulated in terms of two generalized aperture admittance operators, one for each region. These aperture admittance operators are complex and symmetric, just as was the impedance operator in a scattering problem. Hence, characteristic modes can be defined for aperture problems in a manner dual to those defined for scattering problems. These modes have the same desirable properties as the modes in a scattering problem, as follows: a) The characteristic magnetic currents are real (or equiphasal). b) They are weighted orthogonal over the aperture region. c) Their radiation fields (characteristic fields) are Hermitian orthogonal over the radiation sphere. d) Modal solutions for the radiation field converge in a least-squares sense on the radiation sphere.

Aperture problems have been considered by many previous investigators. For an extensive bibliography, see [6]. Small apertures in an infinite conducting plane are usually treated by the Bethe-hole theory [7]. If the small aperture is in a nonplanar surface, the Bethe-hole theory is usually used as an approximation. When the aperture is larger in terms of wavelengths, solution of the appropriate integral equation is usually attempted. The modal solution developed in this paper provides a general approach to the problem, valid for apertures of arbitrary size and shape in conductors of arbitrary size and shape. It reduces to the Bethe-hole theory for small apertures in a conducting plane, and shows how the Bethe-hole theory should be modified for larger apertures and nonplanar surfaces.

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II. Formulation of the Problem

The problem to be considered is the same as that discussed in [5] and illustrated by Fig. 1. It consists of two regions bounded by perfect electric conductors (called conductors for short) and coupled by an aperture. One region, called region a, is considered to be closed, i.e., of bounded extent, and contains impressed sources $\boldsymbol{J}^i, \boldsymbol{M}^i$. The other region, called region b, is considered to be open, i.e., of unbounded extent opened at infinity. The medium in each region is assumed to be loss free, so that the only power loss is due to radiation. We shall develop the theory for the particular case shown in Fig. 1, i.e., region a is closed and region b is open. Slight changes in the interpretation of the theory are required if both regions are open, or if both regions are closed, or if impressed sources exist in both regions.

The equivalence principle [8, sec. 3–5] is used to divide the original problem into two decoupled parts, as shown in Fig. 2. This is accomplished by closing the aperture with a perfect electric conductor (short circuiting the aperture) and placing sheets of magnetic current over the aperture region on both sides of the conductor. In region a, the field is produced by the original sources $\boldsymbol{J}^i, \boldsymbol{M}^i$ plus the equivalent magnetic current sheet

$$\boldsymbol{M} = \boldsymbol{n} \times \boldsymbol{E} \tag{1}$$

over the short-circuited aperture region. In (1), $\boldsymbol{n}$ is the unit normal pointing into region b and $\boldsymbol{E}$ is the unknown electric field in the aperture of the original problem. In region b, the field is produced by the equivalent magnetic current sheet $-\boldsymbol{M}$ over the short-circuited aperture region. The fact that the equivalent magnetic current sheet in region a is $+\boldsymbol{M}$ and that in region b is $-\boldsymbol{M}$ ensures that the tangential component of $\boldsymbol{E}$ is continuous across the aperture in the original problem. The remaining boundary

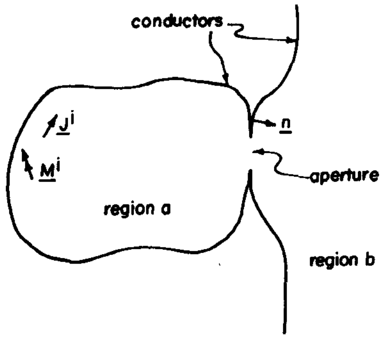

Fig. 1. A typical problem consisting of two regions bounded by conductors and coupled by an aperture.

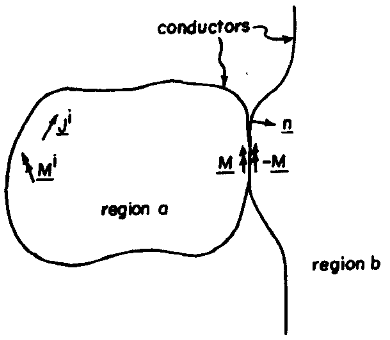

Fig. 2. The original problem decoupled into two equivalent problems by short circuiting the aperture and placing the magnetic current sheets $+\boldsymbol{M}$ and $-\boldsymbol{M}$ over the aperture region.

condition to be satisfied is the tangential component of $\boldsymbol{H}$, which must be continuous across the aperture in the original problem.

The tangential component of the magnetic field in region a over the aperture region, denoted $\boldsymbol{H}_t^a$, is the sum of that due to impressed sources, denoted $\boldsymbol{H}_t^i$, plus that due to the equivalent current $\boldsymbol{M}$, denoted $\boldsymbol{H}_t^a(\boldsymbol{M})$, or

$$\boldsymbol{H}_t^a = \boldsymbol{H}_t^i + \boldsymbol{H}_t^a(\boldsymbol{M}). \qquad (2)$$

Both $\boldsymbol{H}_t^i$ and $\boldsymbol{H}_t^a(\boldsymbol{M})$ are computed in the environment of region a with the aperture short circuited. A similar equation holds for region b except that the equivalent current $-\boldsymbol{M}$ is the only source. Hence, the tangential component of the magnetic field in region b over the aperture region, denoted $\boldsymbol{H}_t^b$, is

$$\boldsymbol{H}_t^b = \boldsymbol{H}_t^b(-\boldsymbol{M}) = -\boldsymbol{H}_t^b(\boldsymbol{M}). \qquad (3)$$

Here, $\boldsymbol{H}_t^b(\boldsymbol{M})$ is computed in the environment of region b with the aperture short circuited. The last equality in (3) is a consequence of the linearity of the operator $\boldsymbol{H}_t^b$. The true solution is obtained when $\boldsymbol{H}_t^a$ of (2) equals $\boldsymbol{H}_t^b$ of (3). The equality can be rearranged to

$$-\boldsymbol{H}_t^a(\boldsymbol{M}) - \boldsymbol{H}_t^b(\boldsymbol{M}) = \boldsymbol{H}_t^i. \qquad (4)$$

This is the basic operator equation for determining the equivalent magnetic current $\boldsymbol{M}$ over the aperture region, or, by (1), the tangential component of $\boldsymbol{E}$ over the aperture in the original problem.

Note that $-\boldsymbol{H}_t^a(\cdot)$ and $-\boldsymbol{H}_t^b(\cdot)$ of (4) have the dimensions of admittance. We define an admittance operator $\boldsymbol{Y}^a$ by

$$\boldsymbol{Y}^a(\cdot) = -\boldsymbol{H}_t^a(\cdot) \qquad (5)$$

and an admittance operator $\boldsymbol{Y}^b$ by

$$\boldsymbol{Y}^b(\cdot) = -\boldsymbol{H}_t^b(\cdot). \qquad (6)$$

Hence, $\boldsymbol{Y}^a$ is the linear operator which when applied to the current sheet $\boldsymbol{M}$ gives the tangential component of $-\boldsymbol{H}$ over the aperture region due to $\boldsymbol{M}$ radiating in the environment of region a with the aperture short circuited. An analogous interpretation applies to the linear operator $\boldsymbol{Y}^b$. The total admittance operator is defined by

$$\boldsymbol{Y}(\cdot) = \boldsymbol{Y}^a(\cdot) + \boldsymbol{Y}^b(\cdot). \qquad (7)$$

Now in admittance operator notation, (4) can be written as

$$\boldsymbol{Y}(\boldsymbol{M}) = \boldsymbol{H}_t^i. \qquad (8)$$

This is the basic equation for the aperture problem.

If impressed sources exist in both regions, say $\boldsymbol{J}^{ia}, \boldsymbol{M}^{ia}$ in region a and $\boldsymbol{J}^{ib}, \boldsymbol{M}^{ib}$ in region b, we interpret $\boldsymbol{H}_t^i$ to be

$$\boldsymbol{H}_t^i = \boldsymbol{H}_t^{ia} - \boldsymbol{H}_t^{ib}. \qquad (9)$$

Here, $\boldsymbol{H}_t^{ia}$ is the tangential component of $\boldsymbol{H}$ over the aperture region due to $\boldsymbol{J}^{ia}, \boldsymbol{M}^{ia}$ radiating in the environment of region a with the aperture short circuited, and $\boldsymbol{H}_t^{ib}$ is the tangential component of $\boldsymbol{H}$ over the aperture region due to $\boldsymbol{J}^{ib}, \boldsymbol{M}^{ib}$ radiating in the environment of region b with the aperture short circuited. However, we emphasize that for the purposes of this paper it is assumed that sources exist only in region a.

III. Properties of the Admittance Operator

We define the symmetric product of two vector functions $\boldsymbol{A}$ and $\boldsymbol{B}$ over the aperture region as

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \iint_{\text{apert}} \boldsymbol{A} \cdot \boldsymbol{B}\, ds. \qquad (10)$$

The product $\langle \boldsymbol{A}^*, \boldsymbol{B} \rangle$, where the asterisk denotes complex conjugate, defines an inner product for the complex Hilbert space of all square integrable functions over the aperture region. The operator $\boldsymbol{Y}^a$ is symmetric, i.e.,

$$\langle \boldsymbol{Y}^a(\boldsymbol{M}_i), \boldsymbol{M}_j \rangle = \langle \boldsymbol{M}_i, \boldsymbol{Y}^a(\boldsymbol{M}_j) \rangle. \qquad (11)$$

This is a statement of reciprocity, proved in [8, sec. 3–8], since in expanded form it is

$$-\iint_{\text{apert}} \boldsymbol{H}_t^a(\boldsymbol{M}_i) \cdot \boldsymbol{M}_j\, ds = -\iint_{\text{apert}} \boldsymbol{H}_t^a(\boldsymbol{M}_j) \cdot \boldsymbol{M}_i\, ds. \qquad (12)$$

Similarly, the operator $\boldsymbol{Y}^b$ is symmetric, i.e.,

$$\langle \boldsymbol{Y}^b(\boldsymbol{M}_i), \boldsymbol{M}_j \rangle = \langle \boldsymbol{M}_i, \boldsymbol{Y}^b(\boldsymbol{M}_j) \rangle \qquad (13)$$

which is a statement of reciprocity for region b. Therefore, the total admittance operator, $\boldsymbol{Y} = \boldsymbol{Y}^a + \boldsymbol{Y}^b$, is symmetric, i.e.,

$$\langle \boldsymbol{Y}(\boldsymbol{M}_i), \boldsymbol{M}_j \rangle = \langle \boldsymbol{M}_i, \boldsymbol{Y}(\boldsymbol{M}_j) \rangle. \qquad (14)$$

However, $\boldsymbol{Y}$ is not self-adjoint with respect to the inner product, i.e.,

$$\langle \boldsymbol{Y}^*(\boldsymbol{M}_i^*), \boldsymbol{M}_j \rangle \neq \langle \boldsymbol{M}_i^*, \boldsymbol{Y}(\boldsymbol{M}_j) \rangle \tag{15}$$

since $\boldsymbol{Y}^a$ and $\boldsymbol{Y}^b$ are not individually self-adjoint. Relationship (14) is a statement of reciprocity, while (15) involves mutual power.

Because $\boldsymbol{Y}$ is symmetric, it follows that the Hermitian adjoint of $\boldsymbol{Y}$ (usually denoted by $\boldsymbol{Y}^*$) is equal to the conjugate of $\boldsymbol{Y}$ (which we denote by $\boldsymbol{Y}^*$). Hence, the Hermitian parts of $\boldsymbol{Y}$ are real, and given by

$$\boldsymbol{G} = \frac{1}{2}(\boldsymbol{Y} + \boldsymbol{Y}^*) \tag{16}$$

$$\boldsymbol{B} = \frac{1}{2j}(\boldsymbol{Y} - \boldsymbol{Y}^*). \tag{17}$$

$\boldsymbol{G}$ is called the conductance operator and $\boldsymbol{B}$ is called the susceptance operator. In terms of its Hermitian parts

$$\boldsymbol{Y} = \boldsymbol{G} + j\boldsymbol{B} \tag{18}$$

which is evident on substitution from (16) and (17). Both $\boldsymbol{G}$ and $\boldsymbol{B}$ are symmetric operators, i.e., (13) is valid for $\boldsymbol{Y}$ replaced by $\boldsymbol{G}$ or $\boldsymbol{B}$, since they are linear combinations of $\boldsymbol{Y}$ and $\boldsymbol{Y}^*$ which are symmetric. Equations (16)–(18) also apply to $\boldsymbol{Y}^a$ and $\boldsymbol{Y}^b$.

Finally, $\boldsymbol{G}$ is a positive definite operator, since for any sheet of magnetic current $\boldsymbol{M}$ we have

$$\begin{aligned}\langle \boldsymbol{M}^*, \boldsymbol{G}(\boldsymbol{M}) \rangle &= \left\langle \boldsymbol{M}^*, \frac{1}{2}(\boldsymbol{Y} + \boldsymbol{Y}^*)(\boldsymbol{M}) \right\rangle \\ &= \frac{1}{2}\langle \boldsymbol{M}^*, \boldsymbol{Y}(\boldsymbol{M}) \rangle + \frac{1}{2}\langle \boldsymbol{M}^*, \boldsymbol{Y}^*(\boldsymbol{M}) \rangle \\ &= \frac{1}{2}\langle \boldsymbol{M}^*, \boldsymbol{Y}(\boldsymbol{M}) \rangle + \frac{1}{2}\langle \boldsymbol{Y}^*(\boldsymbol{M}^*), \boldsymbol{M} \rangle \\ &= \operatorname{Re}\langle \boldsymbol{M}^*, \boldsymbol{Y}(\boldsymbol{M}) \rangle = P_r. \end{aligned} \tag{19}$$

Here, P_r is the sum of the time-average powers radiated by $\boldsymbol{M}$ into regions a and b. For the specific case of Fig. 1, region a is closed and no time-average power is radiated into it. Hence, P_r is that radiated into the open region b. Any magnetic sheet of current external to a bounded conducting surface must radiate some power, however small. This is implied by the work of Saunders [9]. We can think of this as a statement that there are no radiation-free resonances external to a perfect conductor of bounded extent.

IV. Characteristic Aperture Currents

One method to solve an operator equation of the form (8) is to obtain a modal solution in terms of eigenfunctions of the operator. These eigenfunctions can be orthogonal with respect to two operators if we introduce a weight operator into the eigenfunction equation. This weight operator must be positive definite, a property possessed by $\boldsymbol{G}$. Hence, we consider the generalized eigenvalue equation

$$\boldsymbol{Y}(\boldsymbol{M}_n) = y_n \boldsymbol{G}(\boldsymbol{M}_n) \tag{20}$$

where y_n are the eigenvalues and $\boldsymbol{M}_n$ are the eigenfunctions. Let

$$y_n = 1 + jb_n \tag{21}$$

and substitute this and (18) into (20). On cancellation of the common $\boldsymbol{G}(\boldsymbol{M}_n)$ terms, we have

$$\boldsymbol{B}(\boldsymbol{M}_n) = b_n \boldsymbol{G}(\boldsymbol{M}_n). \tag{22}$$

This is equivalent to (20) for determining the eigencurrents $\boldsymbol{M}_n$, and the eigenvalues b_n of (22) are related to the eigenvalues y_n of (20) by (21). Note that (22) is dually related to [3, eq. (13)] used for scattering problems.

We have already shown that both $\boldsymbol{B}$ and $\boldsymbol{G}$ are real symmetric operators. Hence, all eigenvalues b_n are real and all eigenfunctions $\boldsymbol{M}_n$ can be chosen real. (More generally, the $\boldsymbol{M}_n$ can be equiphasal, i.e., a complex constant times a real function, but we will choose them real.) Linearly independent eigenfunctions must satisfy the usual orthogonality relationships

$$\langle \boldsymbol{M}_m, \boldsymbol{G}(\boldsymbol{M}_n) \rangle = 0 \tag{23}$$

$$\langle \boldsymbol{M}_m, \boldsymbol{B}(\boldsymbol{M}_n) \rangle = 0 \tag{24}$$

$$\langle \boldsymbol{M}_m, \boldsymbol{Y}(\boldsymbol{M}_n) \rangle = 0 \tag{25}$$

when $m \neq n$. Furthermore, since the $\boldsymbol{M}_m$ are real, the orthogonality relationships are also valid for Hermitian inner products, i.e., when the $\boldsymbol{M}_m$ is replaced by $\boldsymbol{M}_m^*$ in (23)–(25). It is convenient to normalize the eigencurrents according to

$$\langle \boldsymbol{M}_n, \boldsymbol{G}(\boldsymbol{M}_n) \rangle = 1 \tag{26}$$

i.e., all eigencurrents radiate unit time-average power. Hence, our orthonormalized eigenfunctions obey

$$\langle \boldsymbol{M}_m, \boldsymbol{G}(\boldsymbol{M}_n) \rangle = \langle \boldsymbol{M}_m^*, \boldsymbol{G}(\boldsymbol{M}_n) \rangle = \delta_{mn} \tag{27}$$

$$\langle \boldsymbol{M}_m, \boldsymbol{B}(\boldsymbol{M}_n) \rangle = \langle \boldsymbol{M}_m^*, \boldsymbol{B}(\boldsymbol{M}_n) \rangle = b_n \delta_{mn} \tag{28}$$

$$\langle \boldsymbol{M}_m, \boldsymbol{Y}(\boldsymbol{M}_n) \rangle = \langle \boldsymbol{M}_m^*, \boldsymbol{Y}(\boldsymbol{M}_n) \rangle = y_n \delta_{mn} \tag{29}$$

where δ_{mn} is the Kronecker delta ($\delta_{mn} = 0$, $m \neq n$, and $\delta_{mn} = 1$, $m = n$).

The operators $\boldsymbol{B}$ and $\boldsymbol{G}$ are real and self-adjoint with respect to the symmetric product (10). Hence, we expect that the real set $\{\boldsymbol{M}_n\}$ will be complete in the real Hilbert space of all real square integrable functions $\boldsymbol{M}$ over the aperture region. We also expect $\{\boldsymbol{M}_n\}$ to be complete in the complex Hilbert space of all complex square integrable functions with inner product $\langle \boldsymbol{A}^*, \boldsymbol{B} \rangle$, since a complex function is simply an ordered pair of real functions. Hence, the set of real functions $\{\boldsymbol{M}_n\}$ provides us with a set of real basis functions for expanding the complex equivalent magnetic current $\boldsymbol{M}$. This set simultaneously diagonalizes matrix representations of $\boldsymbol{G}$, $\boldsymbol{B}$, and $\boldsymbol{Y}$. We call $\boldsymbol{M}_n$ the *characteristic currents*, y_n the *characteristic admittances*, and b_n the *characteristic susceptances*. All characteristic conductances $\operatorname{Re}(y_n)$ have been chosen to be unity.

V. Modal Solutions

A modal solution for the equivalent magnetic current over the aperture region can be obtained by using the characteristic currents in Galerkin's method, which is the same as using them for both expansion and testing functions in the method of moments [10]. For this, we take $\boldsymbol{M}$ to be a linear combination of the characteristic currents as

$$\boldsymbol{M}=\sum_n V_n \boldsymbol{M}_n \tag{30}$$

where V_n are coefficients to be determined. Substituting into the original operator equation (8), we obtain

$$\sum_n V_n \boldsymbol{Y}(\boldsymbol{M}_n)=\boldsymbol{H}_t^i. \tag{31}$$

The linearity of the operator $\boldsymbol{Y}$ has been used in (31) to take the Σ and the V_n's outside the $\boldsymbol{Y}$ operation. Next, we take the inner product of (31) with each characteristic current $\boldsymbol{M}_m$ in turn, and obtain

$$\sum_n V_n \langle \boldsymbol{M}_m, \boldsymbol{Y}(\boldsymbol{M}_n)\rangle = \langle \boldsymbol{M}_m, \boldsymbol{H}_t^i\rangle \tag{32}$$

where $m=1,2,\cdots$. (Note that the inner product and symmetric product with respect to $\boldsymbol{M}_m$ are the same, since the $\boldsymbol{M}_m$ are real.) The linearity of the inner product has been used in (32) to take the Σ and the V_n's outside of the inner product. Finally, we use the orthogonality relationship (29) to reduce (32) to

$$V_n y_n = \langle \boldsymbol{M}_n, \boldsymbol{H}_t^i\rangle \tag{33}$$

for $n=1,2,\cdots$. We define the *modal excitation coefficient* for the nth mode to be

$$I_n^i = \langle \boldsymbol{M}_n, \boldsymbol{H}^i\rangle = \iint_{\text{apert}} \boldsymbol{M}_n \cdot \boldsymbol{H}^i\, ds. \tag{34}$$

The subscript t on $\boldsymbol{H}^i$ has been dropped because the dot product with $\boldsymbol{M}_n$ involves only the tangential component. Substituting (34) and (33) into (30), we have

$$\boldsymbol{M}=\sum_n \frac{1}{y_n} I_n^i \boldsymbol{M}_n \tag{35}$$

where $y_n = 1+jb_n$. This is the modal solution to the aperture problem. If the modal currents were not normalized according to (26), the y_n in the denominator of (35) would be replaced by $\langle \boldsymbol{M}_n, \boldsymbol{G}(\boldsymbol{M}_n)\rangle + j\langle \boldsymbol{M}_n, \boldsymbol{B}(\boldsymbol{M}_n)\rangle$.

Note that the modal solution is a special case of the generalized network solution discussed in [5]. In terms of arbitrary expansion and testing functions, the moment solution is [5, eq. (14)]. It is evident that the modal solution diagonalizes the admittance matrix $[Y^a+Y^b]$ according to

$$[Y^a+Y^b]=[\operatorname{diag}(y_n)] \tag{36}$$

where the right-hand side denotes a diagonal matrix with diagonal elements y_n.

VI. Power Through the Aperture

We are considering the particular problem of Fig. 1, where region a is closed and contains all impressed sources $\boldsymbol{J}^i, \boldsymbol{M}^i$, and region b is open. The time-average power into region b is then

$$P_t = \operatorname{Re}\iint_{\text{apert}} \boldsymbol{E}\times\boldsymbol{H}^*\cdot \boldsymbol{n}\, ds = \langle \boldsymbol{M}^*, \boldsymbol{G}^b(\boldsymbol{M})\rangle. \tag{37}$$

Now, since region a is closed, $\langle \boldsymbol{M}^*, \boldsymbol{G}^a(\boldsymbol{M})\rangle = 0$ and $\langle \boldsymbol{M}^*, \boldsymbol{G}^b(\boldsymbol{M})\rangle = \langle \boldsymbol{M}^*, \boldsymbol{G}(\boldsymbol{M})\rangle$. Substituting for $\boldsymbol{M}$ from (30), and using the bilinearity of the symmetric product, we have

$$P_t = \sum_m V_m^* \sum_n V_n \langle \boldsymbol{M}_m, \boldsymbol{G}(\boldsymbol{M}_n)\rangle. \tag{38}$$

The conjugate on $\boldsymbol{M}_m$ has been dropped since $\boldsymbol{M}_m$ is real. Using the orthonormality relationship (27), we reduce the above equation to

$$P_t = \sum_n |V_n|^2. \tag{39}$$

If the $\boldsymbol{M}_n$ were not normalized according to (26), each term in the summation of (39) would be multiplied by $\langle \boldsymbol{M}_n, \boldsymbol{G}(\boldsymbol{M}_n)\rangle$. An alternative form for (39) is obtained by substituting for V_n from (33). The result is

$$P_t = \sum_n |I_n^i / y_n|^2 \tag{40}$$

where I_n^i is the modal excitation coefficient (34). Equation (40) shows explicitly how the transmitted power depends on the characteristic admittances.

VII. Characteristic Fields

The electric field $\boldsymbol{E}_n^a$ and the magnetic field $\boldsymbol{H}_n^a$ produced by a characteristic magnetic current $\boldsymbol{M}_n$ radiating in the environment of region a with the aperture short circuited are called *characteristic fields of region a*. Similarly, the electric field $\boldsymbol{E}_n^b$ and magnetic field $\boldsymbol{H}_n^b$ produced by a characteristic current $-\boldsymbol{M}_n$ radiating in the environment of region b with the aperture short circuited are called *characteristic fields of region b*. The total characteristic fields are defined as

$$\boldsymbol{E}_n = \boldsymbol{E}_n^a + \boldsymbol{E}_n^b \tag{41}$$

$$\boldsymbol{H}_n = \boldsymbol{H}_n^a + \boldsymbol{H}_n^b. \tag{42}$$

Since the operator that gives the field due to a magnetic current is linear, modal solutions of the form (35) can also be written for the fields. These are

$$\boldsymbol{E}=\sum_n \frac{1}{y_n} I_n^i \boldsymbol{E}_n \tag{43}$$

$$\boldsymbol{H}=\sum_n \frac{1}{y_n} I_n^i \boldsymbol{H}_n. \tag{44}$$

They remain valid when superscripts a or b are added to the $\boldsymbol{E}$'s and $\boldsymbol{H}$'s.

For the particular case shown in Fig. 1 (i.e., for region a closed, region b open, and all media loss free), all radiated power must pass through the sphere or portion thereof at infinity in region b, denoted S_∞^b. As stated by (27), the real power radiated by each characteristic current is unity. By reasoning dual to that of [3, sec. III], the characteristic fields are Hermitian orthogonal over S_∞^b. Therefore

$$\frac{1}{\eta}\int_{S_\infty^b} \boldsymbol{E}_m^* \cdot \boldsymbol{E}_n\, ds = \delta_{mn} \tag{45}$$

and

$$\eta \int_{S_\infty^b} \boldsymbol{H}_m^* \cdot \boldsymbol{H}_n \, ds = \delta_{mn}. \tag{46}$$

Hence, the characteristic electric fields $\boldsymbol{E}_n$ are orthogonal with respect to the unweighted Hermitian inner product

$$[\boldsymbol{A}, \boldsymbol{B}] = \iint_{S_\infty^b} \boldsymbol{A}^* \cdot \boldsymbol{B} \, ds. \tag{47}$$

Since the $\{\boldsymbol{M}_n\}$ is complete, the $\{\boldsymbol{E}_n\}$ is also complete in the complex Hilbert space of all radiation fields $\boldsymbol{E}$ producible by currents $\boldsymbol{M}$ over the aperture region, and forms a basis for that space. Similarly, the set of characteristic magnetic fields $\{\boldsymbol{H}_n\}$ is orthogonal with respect to (47), is complete, and forms a basis for the complex Hilbert space of all radiation fields $\boldsymbol{H}$ producible by currents $\boldsymbol{M}$ on the aperture region.

Interpretations of the modal fields in terms of complex power and stored energy can be made in a manner dual to that done for conducting bodies in [3]. In the interest of brevity, we will not derive such formulas here.

VIII. Linear Measurements

Any scalar ρ linearly related to the magnetic current, i.e., a linear function of $\boldsymbol{M}$, will be called a *linear measurement* of $\boldsymbol{M}$. Examples of linear measurements are a) the voltage at some set of terminals, b) the current on some wire, c) a component of electric field at some point in space, and d) a component of magnetic field at some point in space. Every linear functional of $\boldsymbol{M}$ can be written as

$$\rho = \langle \boldsymbol{H}^m, \boldsymbol{M} \rangle \tag{48}$$

where $\boldsymbol{H}^m$ is a given vector function. ρ will be a continuous functional of $\boldsymbol{M}$ if $\boldsymbol{H}^m$ is an ordinary function. If it is desired to include discontinuous functionals of $\boldsymbol{M}$ in the theory, we can let $\boldsymbol{H}^m$ be a symbolic or generalized function.

The determination of $\boldsymbol{H}^m$ has to be a part of the formulation of the problem. For example, if we desire ρ to be the ith component of the magnetic field $\boldsymbol{H}^b$ from $\boldsymbol{M}$ at some point $\boldsymbol{r}$, we would choose $\boldsymbol{H}^m$ to be the magnetic field from a magnetic "test dipole" of unit magnetic moment and i-directed at $\boldsymbol{r}$, calculated in the environment of region b with the aperture short circuited. If we desire ρ to be the ith component of electric field $\boldsymbol{E}^b$ from $\boldsymbol{M}$ at some point $\boldsymbol{r}$, we would choose $\boldsymbol{H}^m$ to be the magnetic field from a unit electric dipole in the negative i direction at $\boldsymbol{r}$, and so on.

We now substitute for $\boldsymbol{M}$ from (35) into (48) and obtain

$$\rho = \sum_n \frac{I_n^i I_n^m}{y_n} \tag{49}$$

where I_n^m is the *modal measurement coefficient*

$$I_n^m = \langle \boldsymbol{H}^m, \boldsymbol{M}_n \rangle = \iint_{\text{apert}} \boldsymbol{H}^m \cdot \boldsymbol{M}_n \, ds. \tag{50}$$

Note that I_n^m is of the same functional form as the modal excitation coefficient (34). Hence, any linear measurement is a symmetric bilinear functional of $\boldsymbol{H}^i$ (the impressed field in the original problem) and $\boldsymbol{H}^m$ (the impressed field in the measurement problem). This symmetry is, of course, a consequence of the fact that $\boldsymbol{Y}$ is a symmetric operator.

IX. Discussion

The eigenvalues of the original operator equation (20) are $y_n = 1 + jb_n$ where b_n are real. These are equal to the modal admittances when the characteristic currents are normalized according to (26). The modal conductances are then unity and the modal susceptances are either positive (capacitive) or negative (inductive). The most important terms in the modal solution (35) are those for which $|y_n|$ are smallest, i.e., $|b_n|$ are smallest. We, therefore, order the modes $n = 1, 2, 3, \cdots$, according to $|b_1| \leqslant |b_2| \leqslant |b_3| \leqslant \cdots$. The Hilbert space of equivalent currents $\boldsymbol{M}$ over the aperture region uses a "power norm"

$$\|\boldsymbol{M}\| = \sqrt{\iint_{\text{apert}} \boldsymbol{M}^* \cdot \boldsymbol{G}(\boldsymbol{M}) \, ds}. \tag{51}$$

Two currents $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ whose difference $\boldsymbol{M}_1 - \boldsymbol{M}_2$ has zero norm are equal because any nontrivial current must radiate some power. The Hilbert space of radiation fields $\boldsymbol{E}$ over the sphere at infinity S_∞^b uses the unweighted Hermitian norm

$$\|\boldsymbol{E}\| = \sqrt{\frac{1}{\eta} \iint_{S_\infty^b} \boldsymbol{E}^* \cdot \boldsymbol{E} \, ds}. \tag{52}$$

This is also a power norm, but no weight operator is required.

For electrically small apertures, only a few modes (normally three or less) are required for accurate solutions. The theory applies to apertures of arbitrary size and shape in conductors of arbitrary size and shape. However, because of the large number of modes which would be needed to compute the equivalent aperture current for large apertures, the theory is primarily useful for apertures of small or intermediate size compared to the wavelength.

The characteristic mode theory gives aperture admittances which include both susceptance and conductance terms. This is in contrast to the usual Bethe-hole theory which describes a small aperture in terms of polarizabilities only. As shown in [11], the polarizabilities determine only the susceptance terms of aperture admittance. A susceptance due to one region can cancel that due to the other region, in which case the Bethe-hole theory predicts infinite power through a small aperture in the loss-free case. It is the conductance term which limits the power, as discussed in [11].

The characteristic mode theory for plane conductors and small apertures reduces to an "augmented" Bethe-hole theory, i.e., to an aperture admittance which includes both susceptance terms and conductance terms. It also shows how the Bethe-hole theory must be modified as the aperture becomes larger and is in conducting surfaces which are curved. It applies to both near fields and far fields if the excitation and measurement vectors are accurately evaluated.

The theory uses a set of real basis functions for all cases. This is in contrast to a modal theory in terms of an unweighted eigenvalue equation, which would require a set of complex basis functions. Not only are real eigenfunctions easier to calculate, but so are the power through the aperture (40) and the radiation field (43). These series converge rapidly when the power norm (51) is used, but slowly when an unweighted norm is used.

References

[1] R. J. Garbacz, "Modal expansions for resonance scattering phenomena," *Proc. IEEE*, vol. 53, pp. 856–864, Aug. 1965.

[2] R. J. Garbacz and R. H. Turpin, "A generalized expansion for radiated and scattered fields," *IEEE Trans. Antennas Propagat.*, vol. AP-19, pp. 348–358, May 1971.

[3] R. F. Harrington and J. R. Mautz, "Theory of characteristic modes for conducting bodies," *IEEE Trans. Antennas Propagat.*, vol. AP-19, pp. 622–628, Sept. 1971.

[4] R. F. Harrington and J. R. Mautz, "Computation of characteristic modes for conducting bodies," *IEEE Trans. Antennas Propagat.*, vol. AP-19, pp. 629–639, Sept. 1971.

[5] R. F. Harrington and J. R. Mautz, "A generalized network formulation for aperture problems," *IEEE Trans. Antennas Propagat.*, vol. AP-24, pp. 870–873, Nov. 1976.

[6] C. M. Butler, Y. Rahmat-Samii, and R. Mittra, "Electromagnetic penetration through apertures in conducting surfaces," *IEEE Trans. Electromagn. Compat.*, vol. EMC-20, pp. 82–93, Feb. 1978.

[7] H. A. Bethe, "Theory of diffraction by small holes," *Phys. Rev.*, vol. 66, pp. 163–182, Oct. 1944.

[8] R. F. Harrington, *Time-Harmonic Electromagnetic Fields*. New York: McGraw-Hill, 1961.

[9] W. K. Saunders, "On solutions of Maxwell's equations in an exterior region," *Proc. Natl. Acad. Sci.*, vol. 38, pp. 342–348, 1952.

[10] R. F. Harrington, *Field Computation by Moment Methods*. New York: Macmillan, 1968. Reprinted by Krieger Publishing Co., Melbourne, FL, 1981.

[11] R. F. Harrington, "Resonant behavior of a small aperture backed by a conducting body," *IEEE Trans. Antennas Propagat.*, vol. AP-30, pp. 205–212, Mar. 1982.

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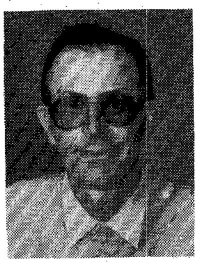

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